Automatic Control Systems

Fundamentals, Theory and Application

Y. N. Sokolov, V. M. Iliushko, A. R. Emad

Edited by Y. N. Sokolov

JAI – Amman

2005

General problems of Automatic Control Theory as a base of aircraft control systems design are considered. The problems of mathematical modeling, stability, accuracy, synthesis, etc. both of the continuous and digital control systems are studied in time- and frequency-domain. Some design and compensation methods of the dynamic systems are presented.

The peculiarity of the book is that it contains not only theoretical material but also abundant number of examples and solutions. For this purpose the high-performance program complex MATLAB is applied in majority sections of the book. Specific feature of examples is that the technology of computer application is considered in each illustration in detail.

The book may be useful for students or researchers in any field of engineering where the dynamic systems are studied.

Published by Jordan Aerospace Industry

Recommended by Science Counsel of National Aerospace University by
N.E. Zhukovsky "Kharkov Aviation Institute", October 23, 2005.

ISBN 0-06-001-J-20

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise, without the prior permission of the publisher. Printed in Jordan.

Contents

Preface…….. 7

Part 1. Continuous Control Systems

1. General concepts of automatic control theory…….. 8
1.1. A feedback control system and general design problem…….. 8
1.2. Example of the control system…….. 10
1.3. History of automatic control development…… 12
2. Fundamentals of the control systems theory…… 14
2.1. Fundamental principles of control……. 14
2.2. Principal control laws…… 16
3. Functional elements of control systems. Classification and general problems……. 20
3.1. Functional elements of control systems…… 20
3.2. Control systems classification…….. 21
3.3. General problems of automatic control theory…….. 22
4. Steady-state characteristics of the elements and continuous control systems…….. 23
4.1. Development of mathematical models of the systems…….. 23
4.2. Steady-state characteristics of the systems…… 24
4.3. Linear approximation…… 24
5. Dynamics of continuous control systems…….. 27
5.1. Ordinary differential equations……. 27
5.2. Operational form of the differential equations…….. 28
5.3. State-vector differential equations……. 29
6. Laplace transformation and transfer function of a system…… 34
6.1. Laplace transformation…….. 34
6.2. Transfer function of a system…… 37
7. Time-domain characteristics of the control systems……. 40
7.1. The standard input signals……. 40
7.2. Transient response and impulse response of a dynamic system... 42
8. Frequency domain characteristics of the control systems……. 47
8.1. Frequency response method…….. 47
8.2. The Fourier transform…….. 47
8.3. Frequency characteristics of the systems and polar plots……. 48
9. Bode diagram…… 52

9.1.	Logarithmic plots……..	52
9.2.	Logarithmic plot of a generalized transfer function……..	54
10.	Transfer function and frequency response of open-loop and closed-loop control systems……..	59
10.1.	Block diagrams……..	59
10.2.	Cascade connection of the blocks…….	59
10.3.	Parallel connection of the blocks……..	60
10.4.	Feedback Connection of the Blocks……..	61
10.5.	Transfer function of a closed-loop system with respect to reference, error and disturbance……..	62
11.	Signal-flow graph method…….	68
11.1.	Signal-flow graph models…….	68
11.2.	Mason's loop rule……..	68
12.	State variable models of control systems……..	73
12.1.	The state vector differential equation……	73
12.2.	The solution of the state vector differential equation……	73
12.3.	Signal-flow graph state model……..	74
12.4.	Determination of the transition matrix……..	76
13.	The stability of linear feedback control systems……..	83
13.1.	The concepts of stability……	83
13.2.	A stability determination of continuous linear control systems by the roots of a characteristic equation……	83
13.3.	Hurwitz stability criterion…….	85
13.4.	Routh-Hurwitz stability criterion……..	87
13.5.	The stability of systems in the time domain……..	93
14.	Stability criteria in the frequency domain…….	94
14.1.	The Nyquist stability criterion……..	94
14.2.	Relative stability and the Nyquist criteria. Margins of stability…	96
14.3.	The logarithmic stability criterion…….	98
14.4.	The stability of control systems with time delays…….	99
15.	Performance of control systems……	103
15.1.	Time-domain performance specifications…….	103
15.2.	Performance of a second-order system…….	103
15.3.	Effect of poles and zeros of higher order system on the performance of the second-order system……..	113
15.4	Performance index and damping ratio……..	120
15.5.	Link between a transient response and roots location on the s-plane……	122

15.6 Steady-state errors of closed-loop control systems……... 123
15.7. Integral performance indices……. 130
15.7.1. Forms of integral performance indices……. 130
15.7.2. Conditions of a minimum of integral performance index and determination of the desirable characteristic polynomial….. 132
15.8. Simplification of linear systems…… 139
16. The design and compensation of feedback control systems in frequency domain and using root locus method…… 140
16.1. Types of compensation……. 140
16.1.1. Characteristics of the phase-lead compensator……. 141
16.1.2. Characteristics of the phase-lag compensator…….. 145
16.2. Compensators design in frequency domain and by means of the root locus method…….. 147
17. Design of optimal control systems in the time domain (modal control)…….. 163
17.1. Statement of the problem…….. 163
17.2. Optimality conditions and the fundamental equations…….. 163

Part 2. Digital Control Systems

18. Advantage of digital control systems…… 170
18.1. Mathematical models of digital control systems, methods of the solution and main properties……. 170
18.1.1. A block diagram and mathematical models of a digital control system…… 170
18.1.2. Difference equations……. 171
18.1.3. The **z**-transform……. 172
18.1.4. Theorems of **z**-transformation……. 174
18.1.5. Inverse **z**-transform……. 175
18.1.6. A method of expansion into a power series…….. 175
18.1.7. A method of partial fractions decomposition and use of the **z**-transform table……. 176
18.1.8. Methods of the solution of difference equations……. 177
18.1.9. A recurrent method of the solution of difference equations…… 177
18.1.10 The solution of difference equations using **z**-transformation…. 182
18.2. Simulation of digital systems…… 183
18.3. A transfer function and a characteristic equation of digital system…… 185

18.4.	State variables equations of digital systems and eigenvalues of a matrix $\boldsymbol{A}$	189
18.5.	The solution of state equations of digital systems	192
18.5.1.	A recurrent method of the solution	192
18.5.2.	The solution of state equations using $\mathbf{z}$-transformation	193
18.6.	Link between state equations and a transfer function of a digital system	197
18.7.	Conversion of the continuous transfer function to a discrete form	199
18.8.	A controllability and an observability of digital systems	202
18.9.	Transfer function and time response of closed-loop digital control system	203
18.9.1.	Determination of a discrete transfer function and transient response of a closed-loop digital control system using block diagram	204
18.9.2.	Determination of a transient response of closed-loop digital system using Simulink model	206
18.9.3.	Determination of a transfer function using a difference equation of closed-loop digital control system	207
19.	The digital control systems stability	210
19.1.	A stability analysis on a $\mathbf{z}$-plane	210
19.2.	Jury stability criterion	213
19.3.	Nyquist stability criterion and bilinear transformation	215
19.4.	Analysis of a digital systems stability using Bode diagram	222
20.	Performance of digital control systems	225
20.1.	Performance of the second-order digital control systems	225
20.2.	An accuracy of digital control systems in steady state	228
20.3.	Relation of a settling time and a phase stability margin	233
21.	Design of analogue and digital PID-controllers	236
21.1.	The purpose of design	236
21.2.	A numerical integration and differentiation	236
21.3.	The block diagram of the digital PID-controller	238
21.4.	Interdependence between gains of the integral and differential components of the PID-controller	248
21.5.	PID-controllers in MATLAB	249
	Selected terms and concepts used in the book	259
	References	265

Examples of the chapters

15. Performance of Control Systems

15.1. Time-Domain Performance Specifications

The time-domain performance specifications are important indices because control systems are inherently time-domain systems. Standard performance measures are usually defined in term of the step response of a system. To analyze the control system performance the next measures are mostly used:

- The percent overshoot, σ_{max}, %;
- The settling time, T_s, sec;
- The pick time, T_p, sec;
- The rise time, T_r, sec;
- The steady-state error, e_{ss};
- The pick value of the time response, h_{max}.

To define these measures the standard test inputs signals are commonly used.

15.2. Performance of a Second-Order System

Consider the response of a second-order system to the unit step signal. The block diagram of the system is shown in fig. 15.1

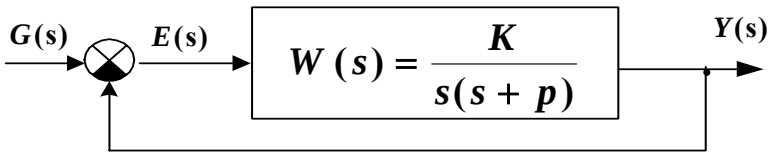


Fig. 15.1. A second-order system

For given system it is possible to write

$$Y(s) = \frac{W(s)}{1 + W(s)} \cdot G(s) = \frac{K}{s^2 + ps + K} \cdot G(s) =$$

$$= \frac{\omega_0^2}{s^2 + 2\xi\omega_0 s + \omega_0^2} \cdot G(s), \qquad \textbf{(15.1)}$$

where p is a pole of a transfer function of an open-loop system; K is a gain; $\omega_0 = \sqrt{K}$ is a natural frequency; $\xi = \frac{p}{2\sqrt{K}}$ is a dimensionless damping ratio. The transfer function of a closed-loop system is

$$\Phi(s) = \frac{Y(s)}{G(s)} = \frac{\omega_0^2}{s^2 + 2\xi\omega_0 s + \omega_0^2}. \qquad (15.2)$$

At a unit step impulse $g(t) = 1(t)$ and therefore, at $G(s) = 1/s$, we will obtain

$$H(s) = \frac{\omega_0^2}{s\left(s^2 + 2\xi\omega_0 s + \omega_0^2\right)}, \qquad (15.3)$$

where $H(s) = Y(s) = \frac{\Phi(s)}{s}$ is the Laplace transform of the transient response. Using the Laplace transform table (Appendix 1) we define a transient response

$$h(t) = L^{-1}[H(s)] = L^{-1}\left[\frac{1}{s}\Phi(s)\right] = 1 - \frac{1}{\beta}e^{-\xi\omega_0 t}\sin(\omega_0\beta t + \theta) =$$

$$= 1 - \frac{1}{\beta}e^{-\frac{t}{\tau}}\sin(\omega_0\beta t + \theta), \qquad (15.4)$$

where $\beta = \sqrt{1-\xi^2}$, $\theta = \arccos\xi$. In expression (15.4) $\tau = 1/\xi\omega_0$ is a time constant of an exponent at which sine wave component fades, and $\omega_0\beta$ is a frequency of sinus-wave oscillations. At $0 < \xi < 1$ process looks like a fading sine wave and a system is sad to be ***underdamped.*** At $\xi = 1$ the system has a ***critical damping***; at $\xi = 0$ a system is ***underdamped***, and at $\xi > 1$ a system is ***overdamped.***

The transfer function (15.2) has two poles:

$$s_{1,2} = -\xi\omega_0 \pm j\omega_0\sqrt{1-\xi^2}. \qquad (15.5)$$

At $\xi > 1$ these poles are real and different, and sinusoidal component is represented by the sum of two exponents, i.e.

$$h(t) = 1 + k_1 e^{-t/\tau_1} + k_2 e^{-t/\tau_2}, \qquad (15.6)$$

where $\tau_1 = 1\Big/\left(\xi\omega_0 + \omega_0\sqrt{\xi^2-1}\right)$ and $\tau_2 = 1\Big/\left(\xi\omega_0 - \omega_0\sqrt{\xi^2-1}\right)$ are time constants of a system.

The transient response of a considered system for various values of a damping ratio ξ at $\omega_0 = 1$ sec $^{-1}$ are shown in fig. 15.2.

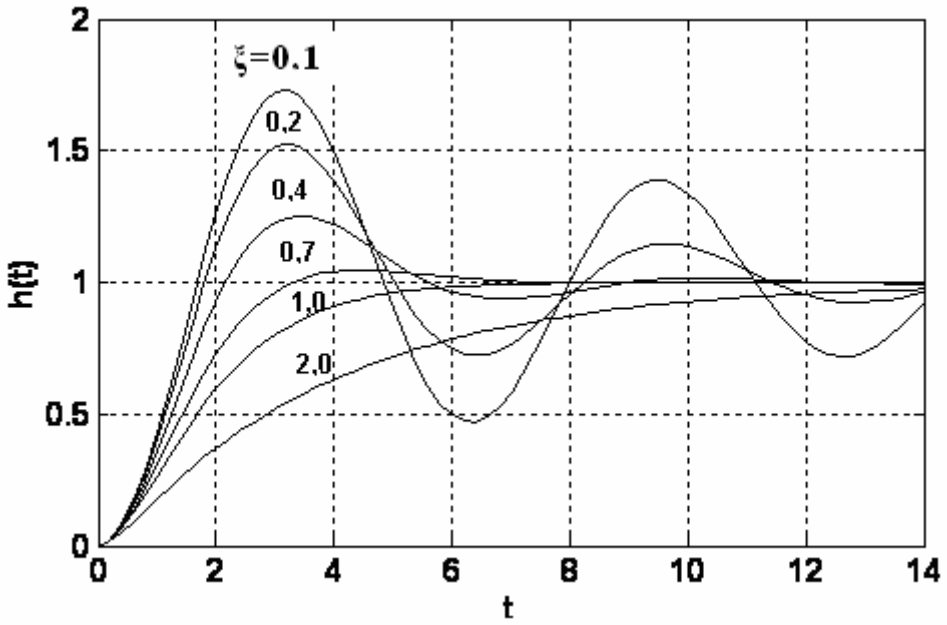


Fig. 15.2. Transient response of a second-order system

Script MATLAB for obtaining these curves by using a transfer function of a closed-loop system (15.2) looks like:

```
>> ksi=[0.1  0.2  0.4  0.7  1.0  2.0];
w0=1;
 t=[0:0.2:14];
for i=1:6
sys=tf([w0^2],[1 2*ksi(i)*w0 w0^2]);
h=step(sys,t);
```

```
Hs(:,i)=h;
end
plot(t,Hs(:,1),t,Hs(:,2),t,Hs(:,3),t,Hs(:,4), t,Hs(:,5),t,Hs(:,6));
xlabel('t'),ylabel('h(t)')
grid
```

The same results can be obtained having a transfer function of an open-loop system and using a MATLAB-function **feedback**:

```
>> ksi=[0.1  0.2  0.4  0.7  1.0  2.0];
w0=1;
t=[0:0.2:14];
for i=1:6
W=tf([w0^2],[1 2*ksi(i)*w0 0]);
sys=feedback(W,[1]);
h=step(sys,t);
Hs(:,i)=h;
end
plot(t,Hs(:,1),t,Hs(:,2),t,Hs(:,3),t,Hs(:,4), t,Hs(:,5),t,Hs(:,6));
xlabel('t'),ylabel('h(t)')
grid
```

Transient response can be computed directly using solution (15.4). As an example we will put relevant script MATLAB and the obtained solution at ξ=0.4 (fig. 15.3):

```
>> t=[0:0.2:14];
ksi= 0.4 ;
w0=1;
beta=sqrt(1-ksi^2);
theta=acos(ksi);
alfa=- ksi*w0*t;
plot(t, 1-(1/beta*exp(alfa)).*sin(w0*beta*t+theta))
xlabel('t'),ylabel('h(t)')
grid
```

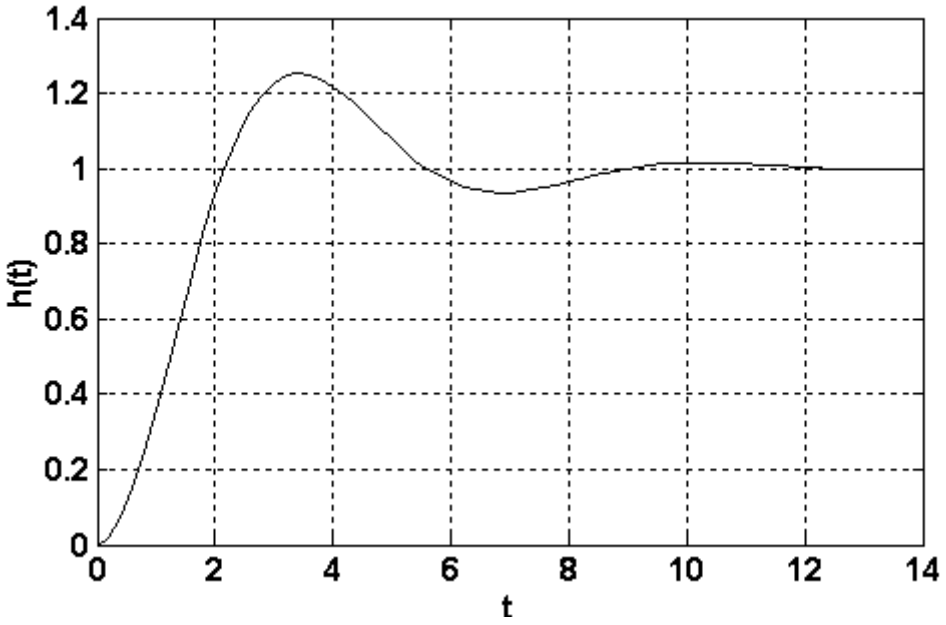


Fig. 15.3. Transient response of a second-order system at ξ=0.4

System response to a δ-impulse, i.e. an impulse transient response, is determined as a derivative from a transient response (15.4) and looks like

$$k(t) = \frac{d\,h(t)}{d\,t} = \frac{\omega_0 t}{\beta} e^{-\xi\omega_0 t} \sin \omega_0 \beta\, t. \qquad \textbf{(15.7)}$$

Plots of impulse transient response (fig. 15.4) can be obtained analogously with simple replacement in an above mentioned script to fig. 15.2 instruction h=**step** (sys, t) by k=**impulse** (sys, t), i.e.:

```
>> ksi=[0.1  0.2  0.4  0.7  1.0  2.0];
w0=1;
t=[0:0.2:14];
for i=1:6
W=tf([w0^2],[1 2*ksi(i)*w0 0]);
sys=feedback(W,[1]);
k=impulse(sys,t);
Ks(:,i)=k;
end
plot(t,Ks(:,1),t,Ks(:,2),t,Ks(:,3),t,Ks(:,4), t,Ks(:,5),t,Ks(:,6));
xlabel('t'),ylabel('k(t)')
grid
```

***Remarks*: for a label of Greek letters in programs MATLAB the set of special symbols TeX (the trade mark of the American Mathematical Society) is used. According to**

TeX, the Greek letter ξ has a label xi. Nevertheless, to avoid possible confusion with indexed variable x, hereinafter we will use a symbol ksi.

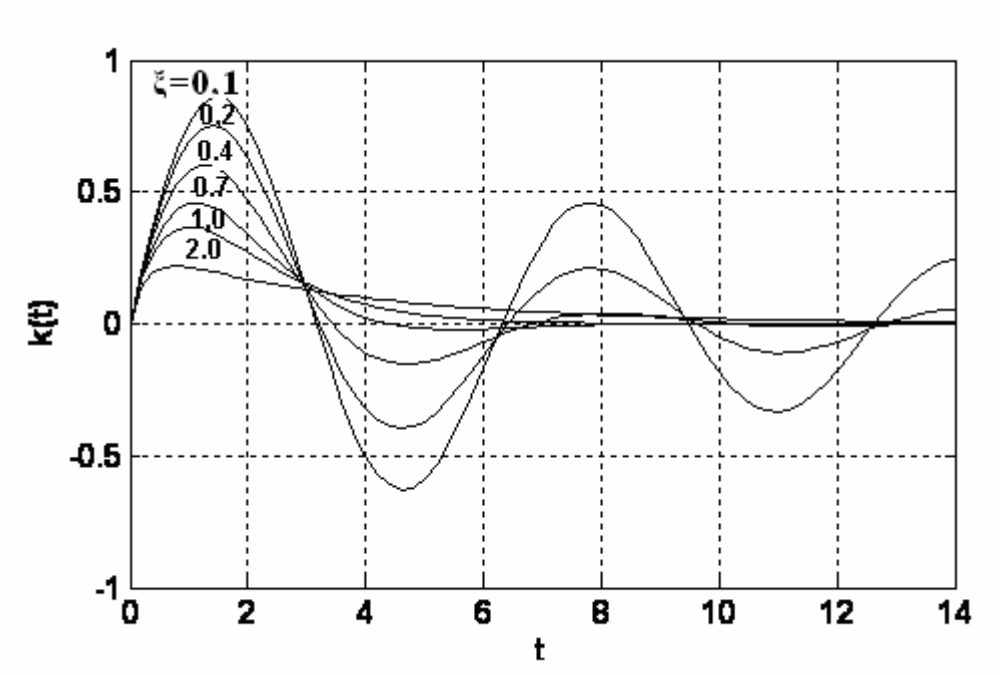


Fig. 15.4. Impulse transient response of a second order system

Standard performance measures are usually defined in terms of the step response of a closed-loop system as shown in fig. 15.5.

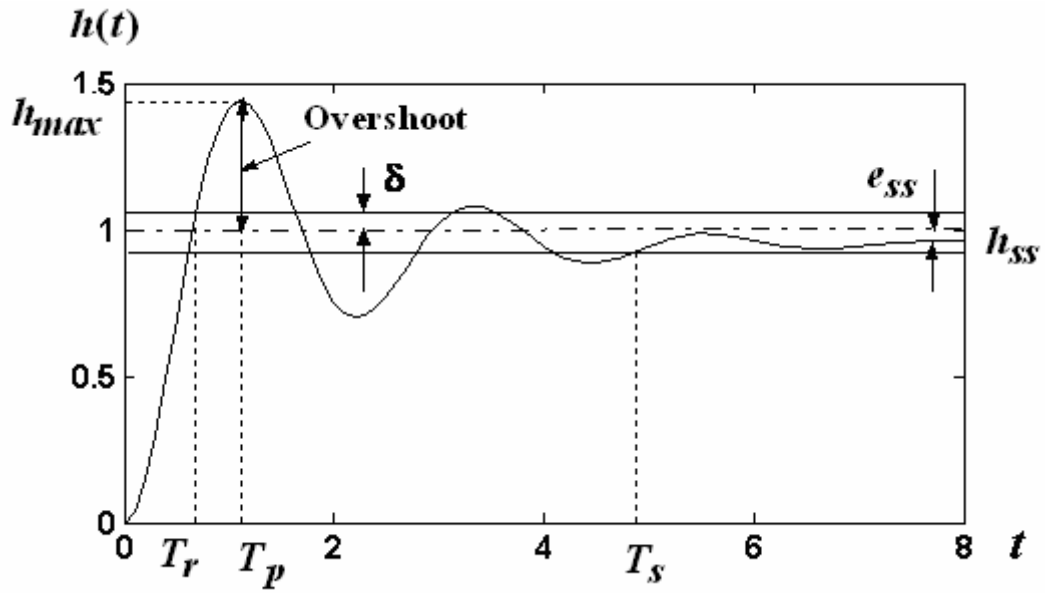


Fig. 15.5. System response to unit step input

The curve in fig. 15.5 is obtained by means of the next program:

```
>>t=[0:0.1:8];
W=tf([20],[2.5  3  1]);
sys=feedback(W,[1]);
```

```
h=step(sys,t);
plot(t,h)
```

The swiftness of the response is measured by the ***rise time*** T_r and ***peak time*** T_p. For underdamped systems with an overshoot, the 0–100 % rise time is useful index.
If the system is overdamped, the 10–90 % rise time is normally used.

The maximum percent overshoot is defined as

$$\sigma_{max} = \frac{h_{max} - h_{ss}}{h_{ss}} \cdot 100\%, \qquad (15.8)$$

where h_{max} is a peak value and h_{ss} is a steady-state value of a response respectively.

The settling time T_S is defined as the time required for the system to settle within a certain percentage δ of the input amplitude. This band of $\pm\delta$ is shown in fig. 15.5.

For the second-order system with closed-loop damping constant $\xi\omega_0$ the transient response remains within 2 % (other requirements are possible, for example, 3 % or 5 %) after four time constant, i.e. $e^{-\xi\omega_0 T_S} < 0.02,$ or $\xi\omega_0 T_S \approx 4$. From this we have

$$T_S \approx 4\tau = \frac{4}{\xi\omega_0}. \qquad (15.9)$$

Therefore we will define the settling time as four time constants τ of the dominant response, where $\tau = 1/\xi\omega_0$ is the time constant relevant to dominating roots (nearest to imagine axis of the s-plane). By the transient response it is possible to determine a steady state error e_{ss}, as shown in fig. 15.5.

To obtain relation of performance measures h_{max} and T_p to parameter ξ, we will take advantage of an impulse transient response (15.7). As it has been noted, it represents a derivative from a transient response (15.4) shown in fig. 15.2. Therefore the first instant (after the beginning of process) at which expression (15.7) equals zero, simultaneously is a time at which tangent to a transient response has a zero declination, i.e. this instant is equal to T_p. From (15.7) follows, that $k(t) = 0$, if $\omega_0\beta T_p = \pi,$ and $\sin\omega_0\beta T_p = \sin\pi = 0,$ where $\beta = \sqrt{1-\xi^2}$. Whence we find

$$T_p = \frac{\pi}{\omega_0 \beta} = \frac{\pi}{\omega_0 \sqrt{1-\xi^2}}. \qquad (15.10)$$

The peak value of a transient response is defined by formula (15.4), having substituted in it T_p from (15.10) and in view of that $\theta = \arccos\xi$, $\tau = 1/\xi\omega_0$, $\omega_0 \beta T_m = \pi$, $\beta = \sqrt{1-\xi^2}$, we will have

$$h_{max} = h(t)\Big|_{t=T_p} = 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\frac{T_p}{1/\xi\omega_0}} \sin(\omega_0 \beta T_p + \arccos\xi) =$$

$$= 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\frac{\pi/\omega_0\beta}{1/\xi\omega_0}} \sin(\pi + \arccos\xi) =$$

$$= 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} \sin(\pi + \arccos\xi).$$

Using angular transformations, we find

$$\sin(\pi + \arccos\xi) = -\sin\arccos\xi = -\sin[\arcsin\sqrt{1-\xi^2}\,] = -\sqrt{1-\xi^2}.$$

Finally we obtain:

$$h_{max} = 1 + e^{-\xi\pi/\sqrt{1-\xi^2}}. \qquad (15.11)$$

As a result, the percent overshoot is defined as

$$\sigma_{max} = 100\, e^{-\xi\pi/\sqrt{1-\xi^2}}. \qquad (15.12)$$

The relation of a percent overshoot σ_{max} and the normalized peak time $\omega_0 T_p = \frac{\pi}{\sqrt{1-\xi^2}}$ versus the damping ratio ξ for a second-order system is shown in fig. 15.6 using equations (15.10) and (15.12) by means of the program:

```
>> ksi=0:0.1:1;
sigma=100*exp(-ksi*pi./sqrt(1-ksi.^2));
omegaT=pi./sqrt(1-ksi.^2);
subplot(1,2,1),plot(ksi,sigma,'-'), xlabel('ksi'), ylabel('sigma'), grid
subplot(1,2,2),plot(ksi,omegaT,'-'), xlabel('ksi'), ylabel('omegaT'), grid
```

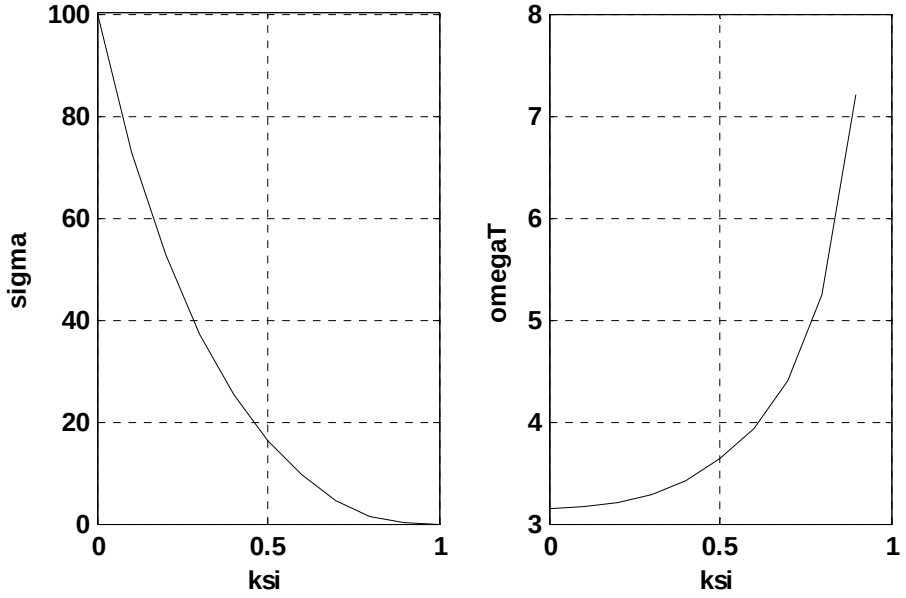


Fig. 15.6. Percent overshoot and the normalized peak time versus the damping ratio ξ for a second-order system

The percent overshoot versus the damping ratio is listed in table 15.1 for selected values of the damping ratio.

Table 15.1

Damping ratio	0.9	0.8	0.7	0.6	0.5	0.4	0.3
Maximum overshoot	0.2	1.5	4.6	9.5	16.3	25.4	37.2

Swiftness of response to a step input depends upon the ξ and ω_0. At the given ξ the more ω_0 the faster response is as shown in fig. 15.7. The maximum overshoot does not depend on ω_0 (that is well visible in fig. 15.7).

Script MATLAB to fig. 15.7:

```
>>ksi=0.2;
w0=[1  10];
t=[0:0.1:5];
for i=1:2
sys=tf([w0(i)^2],[1 2*ksi*w0(i) w0(i)^2]);
y=step(sys,t);
Ys(:,i)=y;
```

```
end
plot(t,Ys(:,1),t,Ys(:,2));
xlabel('t'),ylabel('y(t)')
grid
```

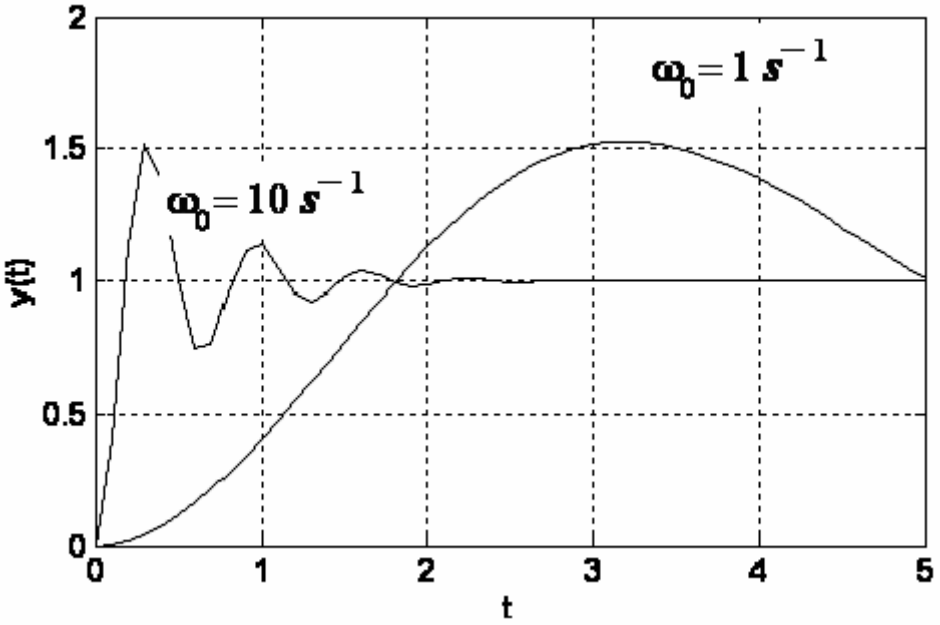


Fig. 15.7. Transient response of the second order system

At the given ω_0 the less ξ the faster response is as shown in fig. 15.8. However it can cause intolerable overshoot.

Script MATLAB to fig. 15.8:

```
>>w0=5;
ksi=[0.7  1];
t=[0:0.05:2.5];
for i=1:2
sys=tf([w0^2],[1 2*ksi(i)*w0 w0^2]);
y=step(sys,t);
Ys(:,i)=y;
end
plot(t,Ys(:,1),t,Ys(:,2));
xlabel('t'),ylabel('y(t)')
grid
```

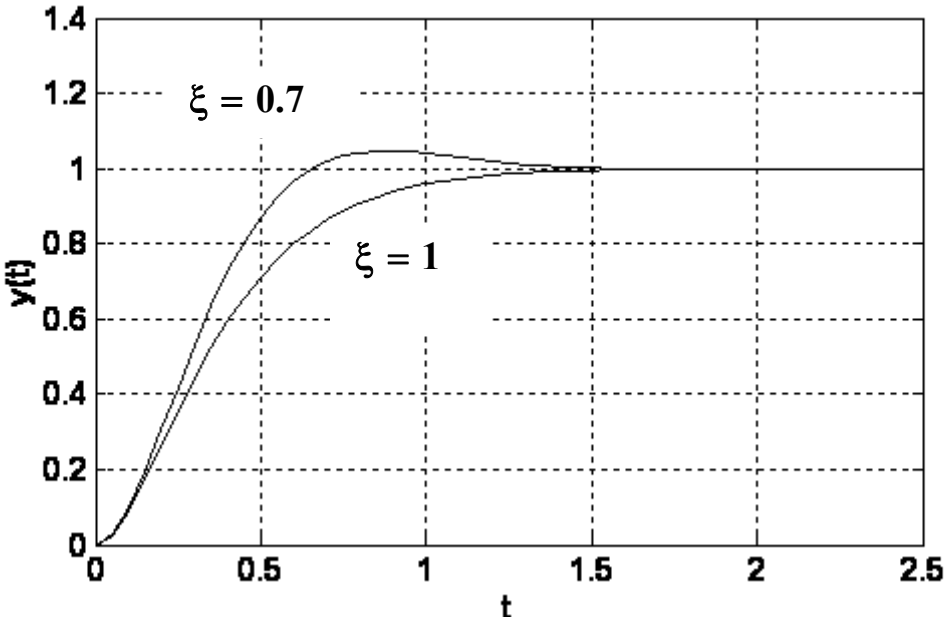


Fig. 15.8. Step response of the second order system

18.2. Simulation of Digital Systems

Let us consider the basic elements used for simulation of systems described by difference equations. We will assume that the unit figured in fig. 18.4, a represents a shift register. The number incomes in the register each T seconds and that number which was stored in it is simultaneously outcomes from it. Let $e(k)$ represents the number entered in the register in the instance $t = kT$. Then in the same instant the number $e(k-1)$ outcomes from the register. The symbolical notation of such memory location is shown in fig. 18.4, b.

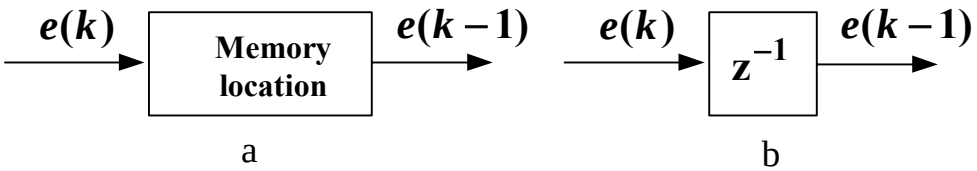


Fig. 18.4. Memory location (shift register)

The combination of such unites including gains and summations blocks allow to build a model of the system described by a difference equation

***Example* 18.9**

Let us consider a difference equation from an example 18.7

$$u(k) = e(k) - e(k-1) - u(k-1).$$

The model of simulation of this equation is shown in fig. 18.5.

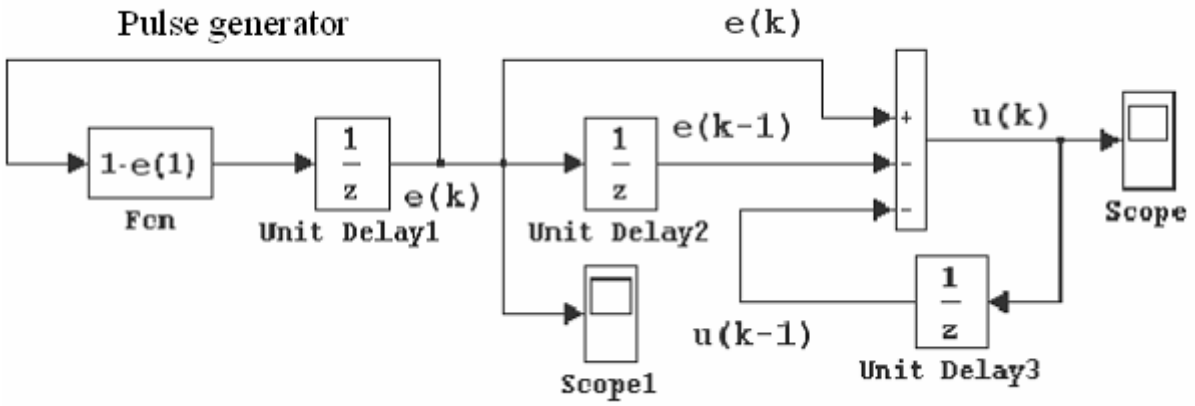
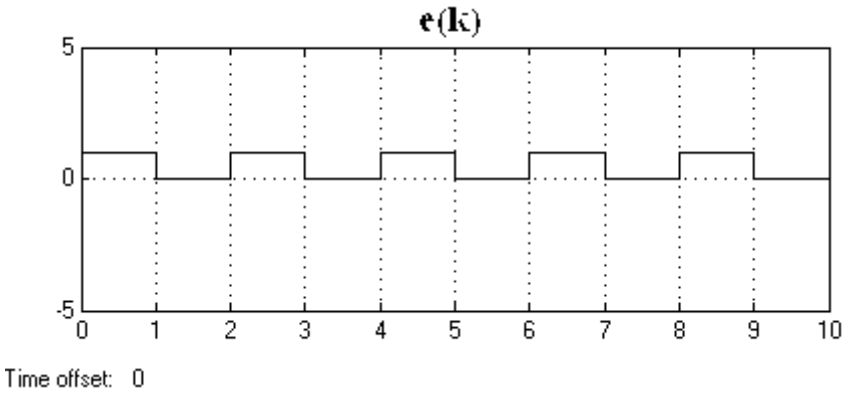


Fig. 18.5. The model of simulation

Block Fcn is in Simulink library (Funcnion & Tables $\rightarrow$ f(t)) and it is used in the pulse generator oscillating series of square waves according to a condition

$$e(k) = \begin{cases} 1, \text{ if } k \text{ is even}, \\ 0, \text{ if } k \text{ is odd} \end{cases}$$

Outcomes of simulation are shown in fig. 18.6.



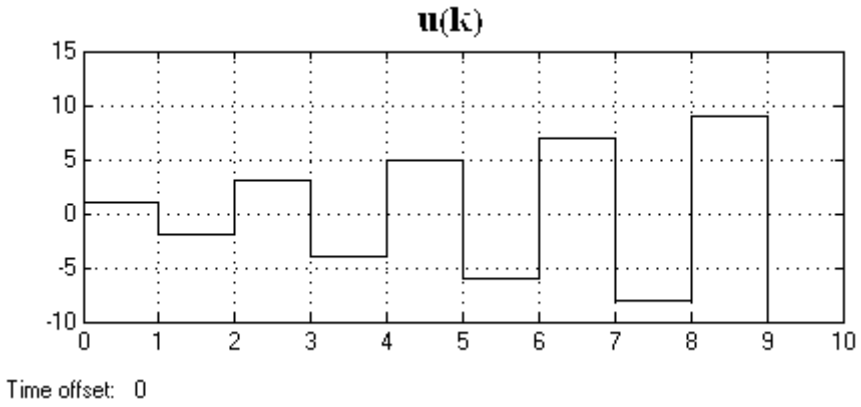


Fig. 18.6. Outcomes of simulation

Comparing the obtained outcomes with outcomes of examples 18.7 and 18.8, we see that they completely coincide.

Let us remind that a key element used at simulation of the continuous systems the integrator with a transfer function $\boldsymbol{s^{-1}}$ is. At simulation of digital systems by such basic unit the delay component having a transfer function $\mathbf{z^{-1}}$ is.